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On the out-of-equilibrium relaxation of the Sherrington–Kirkpatrick model

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Abstract. Starting from a set of assumptions on the long-time limit behaviour of the non-equilibrium relaxation of mean-field models in the thermodynamic limit, we derive analytical results for the long-time relaxation of the Sherrington–Kirkpatrick model, starting from a random configuration.

The system never achieves local equilibrium in any fixed sector of phase space, but remains in an asymptotic out-of-equilibrium regime.

We clearly state and motivate the assumptions made. For the study of the out-of-equilibrium dynamics of spin-glass models, we propose as a tool, both numerical and analytical, the use of 'triangle relations' which describe the geometry of the configurations at three (long) different times.

1. Introduction

In the past few years, most of the study of spin-glass physics has concentrated on the Gibbs–Boltzmann equilibrium measure. As a result of these efforts the mean-field theory is quite well understood [1,2]. The picture that has emerged is one of a phase space with an extremely complex landscape with many minima separated by barriers, some of which are infinitely high. Such divergent barriers lead to ergodicity breaking; a large system is not able to explore the phase space in finite times. For low dimensionalities, the mean field is not exact and the situation is still controversial. In particular, the question of ergodicity breaking and the existence of many pure states is still not settled [3,4].

One of the most striking phenomena observed in the low-temperature phase of real spin glasses is the ageing effect [5,6]: the relaxation of the system depends on its history even after very long times. Although ageing effects seem unusual from the thermodynamical point of view, they have been observed in numerous disordered systems, e.g. in the mechanical properties of amorphous polymers [7], in the magnetic properties of high-temperature superconductors [8], etc. The ageing regime is essentially out-of-equilibrium and, therefore, one has to face the dynamical problem in order to understand most experiments; the study of the Gibbs–Boltzmann weight yields only partial information.

Several phenomenological models have been proposed to account for ageing effects in spin glasses [9]. In particular, a scenario for the basic mechanism of ageing to which we shall refer below has been proposed by Bouchaud [10] (see also the early work of [11]). The main idea is that of 'weak ergodicity breaking', i.e. the system is not allowed to access

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different ergodic components but the relaxation takes place in a rough landscape with 'traps' and the distribution of the 'trapping times' does not have an upper bound.

Still, one would like to have a satisfying microscopic description of the dynamics of spin glasses and, in particular, of the effects mentioned above. In this respect, there are, on one hand, some numerical simulations of realistic systems [12, 13] which yield results in good agreement with experiment. On the other hand, the analytical understanding of the out-of-equilibrium relaxation *in the thermodynamic limit* is much less developed than that of the Gibbs–Boltzmann measure; it is only recently that attention has been paid to this problem.

In [14] it was pointed out that mean-field models exhibit a rich phenomenology in the out-of-equilibrium dynamics, qualitatively similar to those of realistic models and experiments and some analytical results were obtained for a simple model, namely, the p-spin spherical model. One suggestion of that work is that the out-of-equilibrium dynamics of mean-field models can be (at least partially) solvable analytically. The reason for this is the weakness of the long-term memory: the system remembers all its past but in an averaged way; short-time details tend to be washed away by the evolution. Later, numerical analysis has shown that the more standard Sherrington–Kirkpatrick (SK) model also exhibits an asymptotic non-equilibrium regime [15]. Besides, Franz and Mézard [16] have studied the out-of-equilibrium relaxational dynamics of a particle in a random potential in infinite dimensions. They have numerically solved the closed set of mean-field causal equations and have obtained results on the lines of the general picture we shall assume here.

Some years ago, Sompolinsky and Zippelius [17] introduced a dynamical formalism for mean-field spin glasses and used it to study the relaxation within an equilibrium state of the SK model. Later, Sompolinsky [18] proposed to study the equilibrium (Gibbs–Boltzmann measure) of spin-glasses by considering a relaxational dynamics after a very long equilibration time and for large but finite N. The finiteness of N guarantees ergodicity by allowing for the penetration of barriers that diverge in the large-N limit. The existence of divergent barriers led Sompolinsky to postulate a hierarchical set of time scales which were taken as large and eventually went to infinity with N.

However, in a true experimental situation the system is macroscopic and does not reach equilibrium even for very long times. Thus, in order to make contact with the observations, we shall study the relaxational dynamics starting from a random configuration in the thermodynamic limit, i.e. making N infinite from the outset. We shall concentrate on the asymptotics for large times, but throughout this paper we shall understand 'large' as $t \to \infty$ *after* $N \to \infty$ (the opposite order to that considered by Sompolinsky). Under these circumstances, the dynamics is by definition restricted to one ergodic component (not a synonym of 'state' since we are away from equilibrium).

The mean-field dynamical equations then hold rigorously and the solution is *unique* (if one considers the opposite order of the limits, then one has to consider multiple solutions, just as in a system with instantons [19]).

Having already set $N \to \infty$, the mean-field equations of motion have no parameters that become infinite and *a priori* there are no time scales that go to infinity with an external parameter. However, the *solution* to the mean-field equations of motion may exhibit infinite time scales under the conditions of weak ergodicity breaking.

In this scenario, the divergent barriers separating ergodic components are unsurmountable by hypothesis and the system never leaves one ergodic component. However, the landscape has, *within* each ergodic component, traps separated by finite barriers of *all heights* (which are surmountable for long enough times).

As the system ages, it becomes more and more trapped and faces larger and larger barriers simply because it has already crossed the smaller barriers and has had more time to find more 'trapping' traps. In other words, an older system 'sees' a more rugged landscape, of course, not because the actual landscape has changed with time, but because of simple probabilistic reasons. The fact that there is no upper limit to the size of the *finite* barriers makes it possible for this process never to stop and the system never to reach equilibrium.

Consider a two-time function, for example, the autocorrelation function $C(t_w + t, t_w)$. The preceding discussion suggests that the behaviour of the relaxation of the correlation function in terms of t is affected by the overall 'age' t_w. Indeed, it turns out that the 'age' t_w automatically plays a very similar role to the one played by N in the Sompolinsky dynamics: it controls the height of the barriers that are relevant at such times. After a very large time t_w, the system becomes very trapped and any subsequent motion (apart from a fast relaxation inside a trap) takes times that blow-up with t_w. It is clear at this point that if the age of the system is what drives the time scales, then it is essential that the correlation function be *non-homogeneous in time* (i.e. not a function of t exclusively).

Having not maked the assumption of homogeneity in time, we shall obtain that there is no time t_{eq} such that, for all $t_1 > t_2 > t_{eq}$, the two-time (t_1, t_2) functions (e.g. correlation and response functions) obey the equilibrium relations, fluctuation–dissipation theorem (FDT) and homogeneity. This means that the system does not reach equilibrium, not only in the (expected) sense of not reaching the Gibbs–Boltzmann distribution, but also in the wider sense of not reaching any time-independent distribution in a fixed restricted sector of phase space. In other words, the dynamics is different from local equilibrium for all times.

One consequence of these assumptions is that for an infinite system that is rapidly quenched below the critical temperature there is no way to further change the external parameters slower than the internal dynamics of the system, there being no upper time scale. The question of adiabaticity becomes subtle at precisely the critical temperature where the upper time scales change from finite to infinite. One expects that what happens in the adiabatic cooling of an infinite system across the transition temperature is dependent on the nature of the dynamical phase transition [20].

In short, we shall analyse here the out-of-equilibrium dynamics of the SK model basically inspired by the previous results obtained for the simpler p-spin spherical model [14], the numerical simulations of [15] and the phenomenological picture of Bouchaud [10].

The outline of the paper is as follows. In section 2, we review the SK model and its relaxational dynamics. In section 3, we present the assumptions of weak ergodicity breaking and weak long-term memory on which the subsequent treatment is based. In section 4, we discuss the asymptotic equations and their invariances and make two further assumptions suggested by these invariances. Section 5 is devoted to rather general properties of the geometry of the triangles determined by the configuration at three different times and to the discussion of the 'correlation scales'. This last discussion is not particular to the SK model. In sections 6 and 7, we construct the solution to the asymptotic equations.

The results of this paper are by no means exhaustive, although they possess several consequences that can be verified numerically and that are expressed in terms of experimentally measurable quantities. Some of these are presented in section 8 where we also mention some results of [16] relevant to this discussion. Our aim there is only to suggest possible numerical tests which we do not pursue exhaustively in this work. In section 9, we give a qualitative description of our results and contrast them with those previously found in [14] for the p-spin spherical model. Finally, in the conclusions, we summarize our results, we discuss the relationship with Sompolinsky dynamics and we point out some of the many open problems.

2. The relaxational dynamics of the SK model

The SK Hamiltonian is $H = -\sum_{i<j}^{N} J_{ij} s_i s_j$, where the interaction strengths J_{ij} are independent random variables with a Gaussian distribution with zero mean and variance $\overline{J_{ij}^2} = 1/(2N)$. The overbar stands for the average over the couplings. The spin variables take values ± 1. For convenience, we consider a soft-spin version

$$H = -\sum_{i<j}^{N} J_{ij} s_i s_j + a \sum_i (s_i^2 - 1)^2 + \frac{1}{N^{r-1}} \sum_{i_1 < \ldots < i_r}^{N} h_{i_1 \ldots i_r} s_{i_1} \ldots s_{i_r} \tag{2.1}$$

$-\infty \leqslant s_i \leqslant \infty$, $\forall\, i$. Letting $a \to \infty$, one recovers the Ising case, although this is not essential. Additional source terms ($h_{i_1 \ldots i_r}$ time-independent) have been included; if $r = 1$, the usual coupling to a magnetic field h_i is recovered.

The relaxational dynamics is given by the Langevin equation

$$\Gamma_0^{-1} \partial_t s_i(t) = -\beta \frac{\delta H}{\delta s_i(t)} + \xi_i(t) \tag{2.2}$$

where $\xi_i(t)$ is a Gaussian white noise with zero mean and variance $2\Gamma_0$. The mean over the thermal noise is hereafter represented by $\langle \cdot \rangle$.

The mean-field sample-averaged dynamics for $N \to \infty$ is entirely described by the evolution of the two-time correlation and the linear response functions [17]

$$C(t, t') \equiv \frac{1}{N} \sum_{i=1}^{N} \overline{\langle s_i(t) s_i(t') \rangle} \qquad G(t, t') \equiv \frac{1}{N} \sum_{i=1}^{N} \frac{\partial \overline{\langle s_i(t) \rangle}}{\partial h_i(t')}.$$

Following [14], let us introduce the generalized susceptibilities

$$\begin{aligned} I^r(t) &\equiv \lim_{N\to\infty} \frac{r!}{N^r} \sum_{i_1 < \ldots < i_r} \left. \frac{\partial \overline{\langle s_{i_1}(t) \ldots s_{i_r}(t) \rangle}}{\partial h_{i_1 \ldots i_r}} \right|_{h=0} \\ &= r \int_0^t \mathrm{d}t'\, C^{r-1}(t, t') G(t, t') \end{aligned} \tag{2.3}$$

and their generating function $P_d(q)$

$$\lim_{t\to\infty} \lim_{N\to\infty} \left[1 - \frac{r!}{N^r} \sum_{i_1 < \ldots < i_r} \left. \frac{\partial \overline{\langle s_{i_1}(t) \ldots s_{i_r}(t) \rangle}}{\partial h_{i_1 \ldots i_r}} \right|_{h=0} \right] = \int_0^1 \mathrm{d}q'\, P_d(q') q'^r . \tag{2.4}$$

The interest of the function $P_d(q')$ is that if a system reaches equilibrium within a fixed restricted sector of phase space then it is easy to show that $P_d(q')$ should be just a delta function. We shall see that this happens only for $T > T_c$.

3. Weak ergodicity breaking

As in [14], we shall make the following two assumptions, supported by the numerical simulations of this model [15]:

(i) 'Weak' ergodicity breaking

$$\lim_{t\to\infty} C(t,t') = 0 \qquad \forall \text{ fixed } t'$$
$$\frac{\partial C(t,t')}{\partial t} \leqslant 0 \tag{3.1}$$
$$\frac{\partial C(t,t')}{\partial t'} \geqslant 0.$$

This means that the system, after a given time t', starts drifting away (albeit slowly) until it reaches, for sufficiently large times t, the maximal distance $C = 0$ (see section 8). This statement has to be slightly modified in the presence of a magnetic field; in this case '0' has to be substituted by the maximum distance compatible with the remanent magnetization. In particular, this implies that the remanent magnetization $C(t, 0)$ tends to zero in the absence of a magnetic field.

(ii) 'Weak' long-term memory:

$$\lim_{t\to\infty} \int_0^{t'} \mathrm{d}t''\, G(t,t'') = \lim_{t\to\infty} \chi(t,t') = 0 \qquad \forall \text{ fixed } t' \tag{3.2}$$

where $\chi(t,t')$ is the normalized (linear) response at time t to a constant small magnetic field applied from $t' = 0$ up to $t' = t'$, often called the 'thermoremanent magnetization' (see section 8).

This hypothesis is quite crucial since the response function represents the memory the system has of what happened at previous times: the weakness of the long-term memory implies that the system responds to its past in an averaged way; the details of what takes place during a finite time tend to be washed away.

(iii) Finally, we shall make the usual hypothesis that after a (long) time t' there is a quick relaxation in a 'short' time $t - t'$ to some value q, followed by a slower drift away. The parameter q is interpreted in the Sompolinsky–Zippelius dynamics as the Edwards–Anderson parameter for a state [17]. Here the word 'state' certainly does not apply (since a true state is a separate ergodic component) but we may picture q as the size of a 'trap' or the 'width of a channel'. Within these traps the system is fully ergodic while it becomes more and more difficult to escape a trap as time passes. The correlation and response functions are, thus, written in a way that explicitly separates the terms corresponding to the relaxation within a trap

$$C(t,t') = C_{\mathrm{FDT}}(t,t') + \mathcal{C}(t,t')$$
$$G(t,t') = G_{\mathrm{FDT}}(t,t') + \mathcal{G}(t,t').$$

Consistently, $C_{\mathrm{FDT}}(t,t')$ and $G_{\mathrm{FDT}}(t,t')$ are assumed to satisfy the equilibrium relations, i.e. time homogeneity and the FDT

$$C_{\mathrm{FDT}}(t,t') = C_{\mathrm{FDT}}(t-t') \qquad\qquad G_{\mathrm{FDT}}(t-t') = \frac{\partial C_{\mathrm{FDT}}(t-t')}{\partial t'}$$
$$G_{\mathrm{FDT}}(t,t') = G_{\mathrm{FDT}}(t-t')$$

and

$$C_{\mathrm{FDT}}(0) = 1 - q \qquad\qquad \lim_{t-t'\to\infty} C_{\mathrm{FDT}}(t-t') = 0$$
$$\mathcal{C}(t,t) = q \qquad\qquad \lim_{t\to\infty} \mathcal{C}(t,t') = 0.$$

The equilibrium dynamics within a state has been solved with these assumptions [17]. Since this calculation remains the same for the out-of-equilibrium dynamics, although the 'state' must be reinterpreted as a 'trap', we shall not discuss it in this work. We shall concentrate on the evolution of the long-time functions $\mathcal{C}$ and $\mathcal{G}$ and, furthermore, we shall restrict ourselves to the dynamics of the model near and below the critical temperature, $T = T_c - \tau$ with τ small.

4. Asymptotic equations

The full dynamical equations have been written down by Sompolinsky and Zippelius [17]. They are rather cumbersome because, just as in the static case, the spin variables cannot be explicitly integrated away.

Under the assumptions made in the preceding section, i.e. weak ergodicity breaking and weak long-term memory, one can find equations for the evolution of $\mathcal{C}$ and $\mathcal{G}$ *valid asymptotically for large times* $t > t'$ near the transition. They have been presented in [21] and they correspond to the dynamical counterpart of the 'truncated model', the statics of which were solved by Parisi [1]. (The derivation can be achieved in a way that makes the contact with the static free-energy functional near T_c clear by writing the dynamics in the supersymmetric notation [22, 23].) In the absence of a magnetic field, the equations read

$$2(\tau - q)\mathcal{C}(t,t') + y\mathcal{C}^3(t,t') + \int_0^{t'} dt''\, \mathcal{C}(t,t'')\mathcal{G}(t',t'') + \int_0^{t'} dt''\, \mathcal{G}(t,t'')\mathcal{C}(t',t'') + \int_{t'}^{t} dt''\, \mathcal{G}(t,t'')\mathcal{C}(t'',t') = 0 \tag{4.1}$$

$$2(\tau - q)\mathcal{G}(t,t') + \int_{t'}^{t} dt''\, \mathcal{G}(t,t'')\mathcal{G}(t'',t') + 3y\mathcal{C}^2(t,t')\mathcal{G}(t,t') = 0 \tag{4.2}$$

and $y = \frac{2}{3}$. In these equations causality is assumed:

$$\mathcal{G}(t,t') = 0 \qquad \text{for } t < t'. \tag{4.3}$$

These equations do not contain derivatives with respect to time; they have been neglected following the assumption of slow variation of $\mathcal{C}$ and $\mathcal{G}$ for long times.

Evaluating equation (4.2) in $t = t'$ implies either $\mathcal{G}(t,t) = 0$ (corresponding to the high-temperature regime) or

$$2(\tau - q) + 3yq^2 = 0. \tag{4.4}$$

From here, one can obtain the value of the Edwards–Anderson parameter which is also obtained in the static treatment [1].

Even if equations (4.1) and (4.2) are non-local, they can be interpreted as asymptotic *if the crucial assumption of weakness of the long-term memory is made:* equation (3.2) implies that the lower limit $t'' = 0$ in integrals such as

$$\int_0^{t} dt''\, \mathcal{C}(t,t'')\mathcal{G}(t,t'') \tag{4.5}$$

can be substituted by any lower limit $t'' = t_0$ and this has no effect in the integral as long as t and t' both go to infinity.

As has often been noted [11, 18, 21], the asymptotic equations for C and $\mathcal{G}$ have an infinite set of invariances. Indeed, if we perform an arbitrary reparametrization of time

$$\hat{t} = h(t) \qquad \hat{t}' = h(t') \tag{4.6}$$

where h is an increasing function and we redefine

$$\hat{C}(\hat{t}, \hat{t}') = C(h(t), h(t')) \qquad \hat{G}(\hat{t}, \hat{t}') = h'(t')G(h(t), h(t')) \tag{4.7}$$

then the transformed functions $\hat{C}$ and $\hat{G}$ satisfy the same equations in terms of the reparametrized times. This means that, given one solution, we can obtain infinitely many others by reparametrizations.

This invariance is a consequence of having neglected the time derivatives in making the asymptotic limit. The full dynamical equations have no such invariances; because of causality their solution is *unique*. The best we can do with equations (4.1) and (4.2) is find a family of asymptotic solutions (related by reparametrizations). Which one is actually the correct (unique) asymptote can only be decided from equations that do not neglect time derivatives. Throughout this work, we shall try to go as far as possible, keeping the discussion at the reparametrization-invariant level: we shall only obtain solutions *modulo reparametrizations* (however, the relaxation within a trap, as solved in [17] is well determined and not affected by this invariance).

With this in mind, we shall make the following two further assumptions:

(iv) C and $\mathcal{G}$ are related by a reparametrization-invariant formula. Let us first note that because of weak ergodicity breaking without loss we can write

$$\mathcal{G}(t, t') = X[C(t, t'), t'] \frac{\partial C(t, t')}{\partial t'} \theta(t - t') \tag{4.8}$$

which *defines* X. If we now impose that the relation (4.8) be reparametrization invariant, this can only be fulfilled by

$$\mathcal{G}(t, t') = X[C(t, t')] \frac{\partial C(t, t')}{\partial t'} \theta(t - t') \tag{4.9}$$

where X depends only on the times *through* C. Indeed, under reparametrizations (4.6) and (4.7), this equation transforms into

$$\hat{\mathcal{G}}(\hat{t}, \hat{t}') = X[\hat{C}(\hat{t}, \hat{t}')] \frac{\partial \hat{C}(\hat{t}, \hat{t}')}{\partial \hat{t}'} \theta(\hat{t} - \hat{t}') \tag{4.10}$$

i.e. it retains the same form.

Furthermore, if we supplement the definition of $X[z]$ with $X[z] = 1$ for $q < z < 1$ then the relation (4.9) holds for all $C(t, t')$, t' large (cf equation (3.3)). $X[z]$ may be discontinuous in $z = q$ where it jumps from $X[q]$ to 1.

An immediate consequence of equation (4.9) is that all generalized susceptibilities (2.3) are given by

$$I^r(t) = (1 - q^r) + \int_0^q X[q'] \, \mathrm{d}(q'^r). \tag{4.11}$$

Hence, the dynamical generating function of the generalized suceptibilities $P_d(q)$ (equation (2.4)) and, in particular, the asymptotic energy, are entirely determined by the function $X[z]$ and q.

Indeed, assumption (4.9) can also be seen to relate to the fact that the $I^r(t)$ have a finite limit as $t \to \infty$.

(v) Given three large times $t_{\min} \leqslant t_{\rm int} \leqslant t_{\max}$, the corresponding three configurations $s(t_{\max})$, $s(t_{\rm int})$ and $s(t_{\min})$ define a triangle the sides of which are given by the three correlations $C(t_{\rm int}, t_{\min})$, $C(t_{\max}, t_{\min})$ and $C(t_{\max}, t_{\rm int})$. For the three times tending to infinity, we propose that the correlation between the extreme times $C(t_{\max}$ and $t_{\min})$ is completely determined by the correlations between the extreme times and any intermediate time

$$C(t_{\max}, t_{\min}) = f[C(t_{\max}, t_{\rm int}), C(t_{\rm int}, t_{\min})]. \tag{4.12}$$

In other words, we are assuming that when $t_{\min} \to \infty$, the correlation $C(t_{\max}, t_{\min})$ only depends on $t_{\min}$ and $t_{\max}$ through the other two correlations. Note that because of weak ergodicity breaking it would have been *no* assumption to write

$$C(t_{\max}, t_{\min}) = f[C(t_{\max}, t_{\rm int}), C(t_{\rm int}, t_{\min}), t_{\min}]. \tag{4.13}$$

Assumption (v) is equivalent to stating that given a fixed value of $C(t_{\max}, t_{\min}) = C$ allowed by the dynamics, the following limit exists:

$$\lim_{t_{\min}\to\infty/C \text{ fixed}} f[C(t_{\max}, t_{\rm int}), C(t_{\rm int}, t_{\min}), t_{\min}]. \tag{4.14}$$

The relation f is not necessarily smooth and we shall see in the following sections that in fact it is not. We can formally invert this equation by defining the inverse function $\bar{f}$†

$$C(t_{\rm int}, t_{\min}) = \bar{f}[C(t_{\max}, t_{\rm int}), C(t_{\max}, t_{\min})]. \tag{4.15}$$

We are proposing relation (4.12) for the whole range of values of C, including the FDT sector.

Both assumptions made in this section are amenable to numerical checks and we shall discuss them in detail in section 8.

5. Triangle relations

In this section, we shall study the properties of the function f, defined in equation (4.12). *A priori*, f has no reason to be smooth, though we shall assume throughout that it is continuous. All the results we shall present in this section are general, they do not depend on any particular model but just follow from assumption (v) (cf equation (4.12)).

† Note that in the definition of $\bar{f}$, the smallest argument is always on the right.

5.1. Basic properties

The first trivial properties are

$$f(x,1) = f(1,x) = x \tag{5.1.1}$$

which are obtained by choosing $t_{\rm int} = t_{\rm min}$ and $t_{\rm int} = t_{\rm max}$, respectively.

Since we are assuming that the system drifts away at any time, namely, assumption (i)

$$\begin{aligned} f(a,y) &\geqslant f(b,y) \qquad \text{if } a \geqslant b \\ f(y,a) &\geqslant f(y,b) \qquad \text{if } a \geqslant b. \end{aligned} \tag{5.1.2}$$

In particular, for any (a,b),

$$\begin{aligned} f(a,1) = a &\geqslant f(a,b) \\ f(1,b) = b &\geqslant f(a,b) \end{aligned} \tag{5.1.3}$$

and this implies

$$f(a,b) \leqslant \min(a,b). \tag{5.1.4}$$

Consider now four successive times $t_1 < t_2 < t_3 < t_4$. We have

$$\begin{aligned} C(t_4,t_1) &= f[C(t_4,t_3), C(t_3,t_1)] \\ &= f[C(t_4,t_2), C(t_2,t_1)] \\ &= f[C(t_4,t_3), f[C(t_3,t_2), C(t_2,t_1)]] \\ &= f[f[C(t_4,t_3), C(t_3,t_2)], C(t_2,t_1)] \end{aligned} \tag{5.1.5}$$

i.e. f is *associative.*

The existence of a neutral (5.1.1) and the requirement of associativity severely restricts the choice of the function f.

Two important examples of functions (the first one not smooth, the second one smooth) satisfying equations (5.1.1), (5.1.2) and (5.1.5) are

$$f(a,b) = \min(a,b) \tag{5.1.6}$$

the ultrametric relation and

$$f(a,b) = ab. \tag{5.1.7}$$

One can check that this last relation corresponds to the vector s evolving in such a way that the direction of the trajectory at two times is uncorrelated; i.e. which spin flipped at a given time is independent of which spins flipped before. The spherical triangle determined by $s(t_{\rm min})$, $s(t_{\rm int})$ and $s(t_{\rm max})$ is then, for probabilistic reasons, right-angled.

As an example of the physical meaning of the function f, consider the slight variation of equation (5.1.7) which was found in [14] for the long-time correlations C of the p-spin spherical model

$$\frac{C(t_{\rm max}, t_{\rm min})}{q} = \frac{C(t_{\rm int}, t_{\rm min})}{q}\frac{C(t_{\rm max}, t_{\rm int})}{q}. \tag{5.1.8}$$

This can be understood if one defines the 'magnetization' vector for large times t as

$$m_i(t) = \frac{1}{\Delta}\int_0^\Delta \mathrm{d}\mathcal{T}\, s_i(t+\mathcal{T}) \tag{5.1.9}$$

($\Delta \to \infty$, $\Delta/t \to 0$). Then a short computation shows that $(1/N)\sum_i m_i^2 \to q$ and $C(t,t')/q$ is the cosine of the angle subtended by $m(t)$ and $m(t')$ at two widely separated times ($t - t' \gg \Delta$). Then, equation (5.1.8) implies that the magnetization vector $m(t)$ describes a trajectory without memory of the direction and, hence, makes right-angled triangles in any three large (and widely separated) times.

5.2. Correlation scales

Consider now the function $f(a,a)$ satisfying (cf equations (5.1.2))

$$f(a,a) \leqslant a. \tag{5.2.1}$$

The above inequality admits fixed points a_k^* such that $f(a_k^*, a_k^*) = a_k^*$. These points can be isolated or they can form a dense set. In figure 1 we sketch a possible function $f(a,a)$.

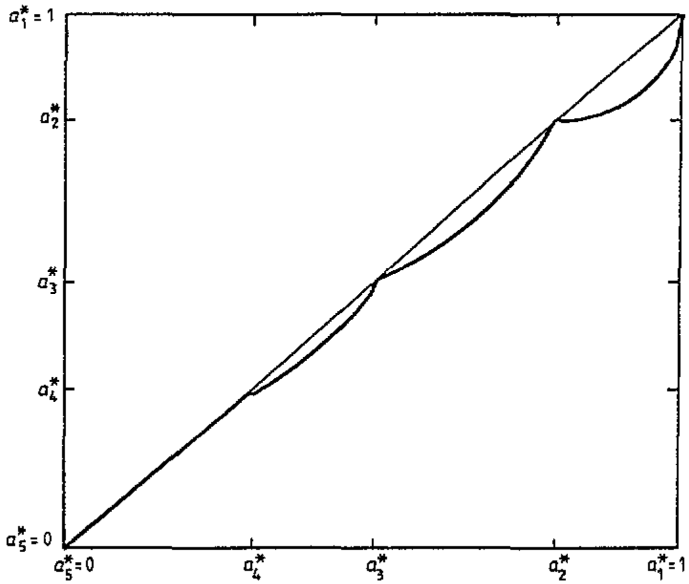

Figure 1. A sketch of the function $f(a,a)$ against a, $a \in [0,1]$.

We now take a succession of (large) times $t_0 < t_1 < \ldots < t_r$, such that the correlation between two successive times is $C(t_{i+1}, t_i) = b$, and compute the correlation $C(t_r, t_0)$ between the two extremes of the succession

$$b^{(r)} \equiv C(t_r, t_0) = f(b, \ldots f(b, f(b,b)) \ldots). \tag{5.2.2}$$

The function f is iterated (r) times and the order of parenthesis is immaterial because of associativity. Choose two correlation values corresponding to consecutive fixed points, say $a_1^* > a_2^*$ (see figure 1), with no other fixed points in between. Then, it is easy to see that, given two values b_1, b_2 with $a_1^* > b_1 > b_2 > a_2^*$, there exists a finite number (s) such that

$$b_1^{(s)} \leqslant b_2 \tag{5.2.3}$$

i.e. with a finite number of steps (iterations of f) with correlation b_1, we can go up to or beyond b_2. In contrast, if we consider a succession of steps each of correlation a^* (a^* is any fixed point), the correlation between the extremes never goes beyond a^* for finite (s).

This suggests that we define *in a reparametrization-invariant way* a correlation scale as the set of correlations that can be connected by relation (5.2.3) for some finite (s). This breaks the whole interval of correlations into equivalence classes.

With this definition, a fixed point a^* constitutes a correlation scale in itself. An interval which is made up of a dense set of fixed points such as (a_5^*, a_4^*) in figure 1 is then a dense set of scales.

The interval of correlations (none of them a fixed point of $f(a, a)$) contained between two fixed points is a correlation scale which we shall call 'discrete'.

Note that the time difference $(t_{i+1} - t_i)$ needed to achieve a certain correlation is not independent of the time t_i, but we can suppose that it does not decrease with time, i.e.

$$C(t_{i+1}, t_i) = C(t_i, t_{i-1}) \Rightarrow t_{i+1} - t_i \geqslant t_i - t_{i-1}. \tag{5.2.4}$$

Then, the above definition of correlation scales translates into a definition of 'infinite time scales'.

5.3. *Ultrametric relations*

Let us start by showing that the relation between fixed points is ultrametric. Let a_1^*, a_2^* be any two fixed points with $a_1^* > a_2^*$

$$a_2^* = f(a_2^*, a_2^*) \leqslant f(a_2^*, a_1^*) \leqslant a_2^*$$

$$a_2^* = f(a_2^*, a_2^*) \leqslant f(a_1^*, a_2^*) \leqslant a_2^*.$$

We have used here equations (5.1.2) and (5.1.4). Hence, for any two fixed points

$$f(a_1^*, a_2^*) = f(a_2^*, a_1^*) = \min(a_1^*, a_2^*) \tag{5.3.1}$$

and ultrametricity holds.

Next, we consider a discrete scale limited by a_i^* and a_{i+1}^*, two consecutive fixed points and a number b such that $a_i^* > b > a_{i+1}^*$. We assume that f is a smooth function of x, y within the scale. Then, one has (see appendix A)

$$f(a_i^*, b) = (b, a_i^*) = b \qquad \forall\, b \in (a_{i+1}^*, a_i^*)$$

$$f(a_{i+1}^*, b) = f(b, a_{i+1}^*) = a_{i+1}^* \qquad \forall\, b \in (a_{i+1}^*, a_i^*)$$

i.e. a_i^* acts as the *neutral* and a_{i+1}^* as the 'zero' inside the scale.

Consider now two different scales limited, respectively, by $a_i^* > a_{i+1}^*$ and $a_k^* > a_{k+1}^*$. Using the previous result, we now show that the relation $f(b_1, b_2)$ between two correlations belonging to each scale $a_i^* \geqslant b_1 \geqslant a_{i+1}^*$ and $a_k^* \geqslant b_2 \geqslant a_{k+1}^*$ is ultrametric

$$f(b_1, b_2) = f(b_2, b_1) = \min(b_1, b_2). \tag{5.3.2}$$

Let us start by choosing $b_1 = a_{i+1}^*$ and $a_{i+1}^* > a_k^*$, then

$$f(f(b_2, a_k^*), a_{i+1}^*) = f(b_2, a_{i+1}^*) = f(b_2, f(a_k^*, a_{i+1}^*)) = f(b_2, a_k^*) = b_2. \tag{5.3.3}$$

Hence

$$f(b_2, a_{i+1}^*) = b_2 \tag{5.3.4}$$

and similarly

$$f(a_{i+1}^*, b_2) = b_2. \tag{5.3.5}$$

Now choose b_1 satisfying $a^*_{i+1} < b_1 \leqslant a^*_i$, then

$$f(b_2, b_1) = f(f(b_2, a^*_k), b_1) = f(b_2, f(a^*_k, b_1)) = f(b_2, a^*_k) = b_2. \quad (5.3.6)$$

Hence

$$f(b_2, b_1) = b_2 \quad (5.3.7)$$

and similarly

$$f(b_1, b_2) = b_2. \quad (5.3.8)$$

Within a scale, ultrametricity does not hold, but if we assume that there f is smooth, it is then a one-dimensional formal group law and we have for b_1, b_2 within the kth scale that $f(b_1, b_2)$ is of the form

$$f(b_1, b_2) = \jmath_k^{-1}[\jmath_k(b_1) \cdot \jmath_k(b_2)] \quad (5.3.9)$$

for some function $\jmath_k(x)$ which can be different for each scale. This is a well known result from formal group theory [24] and we present it in appendix B.

Furthermore, this implies that within a scale (see equation (B.10))

$$C(t, t') = \jmath_k^{-1}\left(\frac{h_k(t')}{h_k(t)}\right) \quad (5.3.10)$$

for some increasing function $h_k(x)$ (appendix B).

6. Dynamical equations

In this section, we shall study the asymptotic dynamical equations of the SK model (equations (4.1) and (4.2)). We shall start by simplifying the equations using assumption (iv).

Let us define two functionals

$$F[C] = -\int_C^q \mathrm{d}C'\, X[C'] \quad (6.1)$$

$$H[C] = -\int_C^q \mathrm{d}C'\, C'^2 X[C']. \quad (6.2)$$

Note that even if X is discontinuous, F and H are continuous. Thus, inserting equation (4.9) in equations (4.1) and (4.2), we get

$$-3yq^2 C(t, t') + yC^3(t, t') + \int_0^{t'} \mathrm{d}t''\, C(t, t'') \frac{\partial F[C(t', t'')]}{\partial t''} + \int_0^{t'} \mathrm{d}t''\, \frac{\partial F[C(t, t'')]}{\partial t''} C(t', t'')$$
$$+ \int_{t'}^{t} \mathrm{d}t''\, \frac{\partial F[C(t, t'')]}{\partial t''} C(t'', t') = 0 \quad (6.3)$$

$$-3yq^2 \frac{\partial F[C(t, t')]}{\partial t'} + \int_{t'}^{t} \mathrm{d}t''\, \frac{\partial F[C(t, t'')]}{\partial t''} \frac{\partial F[C(t'', t')]}{\partial t'} + 3y \frac{\partial H[C(t, t')]}{\partial t'} = 0. \quad (6.4)$$

This last equation can be integrated once with respect to t' to give

$$-3yq^2F[C(t,t')] + \int_{t'}^{t} dt'' \frac{\partial F[C(t,t'')]}{\partial t''} F[C(t'',t')] + 3yH[C(t,t')] = 0. \tag{6.5}$$

We now have two reparametrization-invariant equations for $C(t,t')$ which have to be satisfied simultaneously. Equations (6.3) and (6.5) can be written in a form in which the times disappear using assumption (v). Indeed, with definition (4.15) they formally become

$$-3yq^2C + \int_0^C dC'\, X[C']\bar{f}(C,C') - \int_0^C dC'\, F[\bar{f}(C,C')] + \int_C^q dC'\, X[C']\bar{f}(C',C) + yC^3 = 0 \tag{6.6}$$

$$-3yq^2F[C] + \int_C^q dC'\, X[C']F[\bar{f}(C',C)] + 3yH[C] = 0 \tag{6.7}$$

since $\bar{f}(C,C')$ is not necessarily a well behaved function over the whole interval [0, 1] and we shall take care of this point in the next subsection. Equations (6.6) and (6.7) together determine $X[C]$ and $\bar{f}(C,C')$. Having eliminated the explicit time dependence, we have effectively divided by the reparametrization group. In what follows, we shall concentrate on studying the solution to these equations.

6.1. Equations within a discrete scale

Let us take C belonging to a discrete scale $C \in (a_2^*, a_1^*)$. Due to the ultrametricity between different scales (cf equation (5.3.2)), we can use

$$\bar{f}(C,C') = C' \tag{6.1.1}$$

when C is outside the scale of C'. This allows us to simplify equations (6.6) and (6.7) and, furthermore, deal with a region where the function $\bar{f}(C,C')$ is smooth. In these regions, $\bar{f}(C,C')$ can be written as (see appendix B)

$$\bar{f}(C,C') = \jmath^{-1}\left(\frac{\jmath(C')}{\jmath(C)}\right) \qquad \text{for } C \geqslant C' \tag{6.1.2}$$

and since $\jmath(a_1^*) = 1$

$$\bar{f}(a_1^*, C) = C. \tag{6.1.3}$$

Therefore, the equations become

$$-3yq^2C + a_2^*F[a_2^*] - CF[a_1^*] + yC^3 - 2\int_0^{a_2^*} dC'\, F[C'] + \int_{a_2^*}^{C} dC'\, X[C']\bar{f}(C,C')$$
$$- \int_{a_2^*}^{C} dC'\, F[\bar{f}(C,C')] + \int_C^{a_1^*} dC'\, X[C']\bar{f}(C',C) = 0 \tag{6.1.4}$$

$$-3yq^2F[C] + 3yH[C] - F[C]F[a_1^*] + \int_C^{a_1^*} dC'\, X[C']F[\bar{f}(C',C)] = 0. \tag{6.1.5}$$

Evaluating equation (6.1.5) in $\mathcal{C} = a_1^*$ and $\mathcal{C} = a_2^*$, respectively, we have

$$3yH[a_1^*] = F[a_1^*](F[a_1^*] + 3yq^2) \qquad (6.1.6)$$

$$3yH[a_2^*] = F[a_2^*](F[a_2^*] + 3yq^2). \qquad (6.1.7)$$

Moreover, differentiating equation (6.1.5) with respect to $\mathcal{C}$ and evaluating in $\mathcal{C} = a_1^*$, we obtain

$$X[a_1^*](-3yq^2 - 2F[a_1^*] + 3ya_1^{*2}) = 0 \qquad (6.1.8)$$

and this implies $X[a_1^*] = 0$ or

$$F[a_1^*] = \frac{3y}{2}(a_1^{*2} - q^2). \qquad (6.1.9)$$

Differenting equation (6.1.5) twice with respect to $\mathcal{C}$ and evaluating in $\mathcal{C} = a_1^*$, we have

$$X[a_1^*]\left(6ya_1^* - X[a_1^*]\frac{\partial \bar{f}(\mathcal{C}', \mathcal{C})}{\partial \mathcal{C}}\bigg|_{\mathcal{C}'=\mathcal{C}=a_1^*}\right) = 0. \qquad (6.1.10)$$

The derivative on the right-hand side can be calculated using equation (6.1.2) and is equal to one. Then, if $X[a_1^*] \neq 0$

$$X[a_1^*] = 6ya_1^*. \qquad (6.1.11)$$

6.2. *X within a discrete scale*

We shall assume that:

(vi) X is a non-decreasing function.

Under this last assumption, we shall show that X is constant within a discrete scale [25]. Differentiating equations (6.1.4) and (6.1.5) with respect to $\mathcal{C}$, multiplying the first one by $X[\mathcal{C}]$ and subtracting, we obtain

$$\int_{a_2^*}^{\mathcal{C}} \mathrm{d}\mathcal{C}'\, X[\mathcal{C}]\frac{\partial \bar{f}(\mathcal{C}, \mathcal{C}')}{\partial \mathcal{C}}(X[\bar{f}(\mathcal{C}, \mathcal{C}')] - X[\mathcal{C}']) + \int_{\mathcal{C}}^{a_1^*} \mathrm{d}\mathcal{C}'\, X[\mathcal{C}']\frac{\partial \bar{f}(\mathcal{C}', \mathcal{C})}{\partial \mathcal{C}}(X[\bar{f}(\mathcal{C}', \mathcal{C})] - X[\mathcal{C}]) = 0. \qquad (6.2.1)$$

Since this equation is valid within any scale, we see that a sufficient condition for equations (6.1.4) and (6.1.5) to be compatible is for X to be constant within each scale. Let us now show that if X is non-decreasing, this is the only possibility. Differentiating equation (6.2.1) with respect to $\mathcal{C}$ and then evaluating in $\mathcal{C} = a_1^*$ yields

$$X[a_1^*]\int_{a_2^*}^{a_1^*} \mathrm{d}\mathcal{C}'\left(\frac{\partial \bar{f}(\mathcal{C}, \mathcal{C}')}{\partial \mathcal{C}}\right)^2\bigg|_{\mathcal{C}=a_1^*} X'[y] = 0. \qquad (6.2.2)$$

This equation admits the solution $X[a_1^*] = 0$, which corresponds to the high-temperature phase. If this is not the case and assuming that $X'[z] \geqslant 0$, the integrand should vanish. The squared factor looks like

$$\frac{\partial \bar{f}(\mathcal{C}, \mathcal{C}')}{\partial \mathcal{C}}\bigg|_{\mathcal{C}=a_1^*} = \jmath'(a_1^*)\frac{\jmath(\mathcal{C}')}{\jmath'(\mathcal{C}')} = \frac{\partial \ell(z)}{\partial z}\bigg|_{z=a_1^*}\left(\frac{\partial \ell(\mathcal{C}')}{\partial \mathcal{C}'}\right)^{-1} \qquad (6.2.3)$$

(see appendix B) and does not vanish for $\mathcal{C}' > a_2^*$. Thus,

$$X'[y] = 0 \qquad \Rightarrow \qquad X[y] = X \text{ constant} \qquad (6.2.4)$$

for $y \in (a_2^*, a_1^*)$. Hence, using equation (6.1.11)

$$X[y] = X = 6ya_1^*. \qquad (6.2.5)$$

6.3. No discrete scales

Let us show that the discrete scales collapse. Using the fact that X is constant within the discrete scale, we have

$$F[a_2^*] = F[a_1^*] + X(a_2^* - a_1^*) \tag{6.3.1}$$

$$H[a_2^*] = H[a_1^*] + \tfrac{1}{3}X(a_2^{*3} - a_1^{*3}). \tag{6.3.2}$$

Putting this into equations (6.1.6) and (6.1.7) and using equation (6.1.9) we have

$$(a_2^* - a_1^*)^2[y(a_2^* + 2a_1^*) - X] = 0. \tag{6.3.3}$$

This equation gives, as one possible root, $a_2^* = 4a_1^*$, which is not acceptable since, by hypothesis, $a_2^* \leqslant a_1^*$. Hence, we are left with

$$a_2^* = a_1^* \tag{6.3.4}$$

and this implies that each discrete scale is indeed empty.

Having shown that there are no discrete scales (except for the FDT scale, which is not contained in the previous equations), one sees that the solution should verify ultrametricity for all values of the correlations C.

It is interesting to remark that the solution for the asymptotic dynamics of the p-spin spherical model presented in [14] can be easily obtained using this formalism. Indeed, the same steps followed in this section imply to the case $a_2^* = 0$ and $a_1^* = q$: there is only one discrete scale (apart from the FDT scale) and the solution has the form of equation (5.3.10) [26].

7. Ultrametric solution

Let us now describe in detail the ultrametric solution. For $t_{\min} \to \infty$,

$$C(t_{\max}, t_{\min}) = \min(C(t_{\max}, t_{\text{int}}), C(t_{\text{int}}, t_{\min})) \qquad \text{if } C(t_{\max}, t_{\min}) \leqslant q \tag{7.1}$$

and if $C(t_{\max}, t_{\min}) \geqslant q$ then the Sompolinsky–Zippelius solution holds.

In the preceding section, we concluded that we have a dense set of scales, so that

$$\bar{f}(C, C') = C' \qquad \forall\, C' < C. \tag{7.2}$$

Thus, equations (6.6) and (6.7) simplify to

$$-3yq^2C - CF[C] + yC^3 + \int_0^C \mathrm{d}C'\, C'X[C'] - \int_0^C \mathrm{d}C'\, F[C'] = 0 \tag{7.3}$$

$$-3yq^2F[C] - (F[C])^2 + 3yH[C] = 0 \tag{7.4}$$

and from here it follows that equations (6.1.6) and (6.1.9) hold for every value $C < q$:

$$3yH[C] = F[C](F[C] + 3yq^2)$$

$$F[C] = \frac{3y}{2}(C^2 - q^2). \tag{7.5}$$

Differentiating equation (7.5) with respect to C, we obtain

$$X(C) = 3yC. \tag{7.6}$$

For $P_d(q)$, this yields (cf equations (2.3), (2.4) and (4.11))

$$P_d(q') = (1 - X[q'])\delta(q' - q) + 3yU(q') \tag{7.7}$$

where $U(q') = 1$ if $0 < q' < q$, and zero otherwise. The value of q is given by equation (4.4). Hence, we have found that

$$P_d(q') = P(q') \tag{7.8}$$

for the SK model, where $P(q')$ is the Parisi functional order parameter associated with the Gibbs–Boltzmann measure [1], also implying that the dynamic and static transition temperatures coincide. This equality is not obvious and is a property of this particular model. Indeed, this same dynamics yields for the model of [14] a dynamic $P_d(q)$ which is *different* from the static one. For the SK model, the energy and susceptibility *to leading order in N* coincide with the corresponding equilibrium values and the size of the 'traps' encountered for large times coincide with the size of the equilibrium states.

In particular, ultrametricity implies that a plot of $C(t, t')$ against t' tends to have a long plateau. More precisely, consider the function

$$\hat{C}(\mu) = \lim_{t\to\infty} C(t, \mu t). \tag{7.9}$$

It is easy to see that equation (7.1) implies that $\hat{C}(\mu)$ drops from one to a certain value $\tilde{q}$ $(0 \leqslant \tilde{q} \leqslant q)$ in a small neighbourhood of $\mu = 1$, remains constant and equal to $\tilde{q}$ in the interval (0, 1) and drops from $\tilde{q}$ to zero in a small neighbourhood of $\mu = 0$. The actual value of $\tilde{q}$ cannot be determined unless one goes beyond the reparametrization-invariant results. This last result was verified in [16] for the model studied there.

8. Simulations and measurable results

In this section, we discuss some consequences of the assumptions and derivations of the previous sections that can be checked with numerical simulations. Some of these involve magnitudes that are measurable experimentally; for a toy model such as the SK this is not such an advantage, but it would be desirable if some of these results also turn out to hold for finite-dimensional systems.

Our aim here is not to make an exhaustive numerical analysis but to propose some checks that can be useful in the study of the out-of-equilibrium dynamics of spin glasses.

Our results have been obtained in the thermodynamic limit for asymptotic times $\lim_{t\to\infty}$ after $\lim_{N\to\infty}$. In a simulation, this means that one has to eliminate finite-size effects.

First, the two initial assumptions (i) and (ii) have already been observed in [15], where the out-of-equilibrium dynamics of the hypercube spin glass has been studied numerically. This model is expected to reproduce the SK model for high dimensionalities [27, 28].

Assumption (i), weak ergodicity breaking, has been verified by plotting the correlation function $C(t + t_w, t_w)$ against t in a log–log plot for different waiting times t_w (figure 3 in [15]). Furthermore, numerical simulations also support this assumption in the realistic 3D Edwards–Anderson model [13].

Assumption (ii), weakness of the long-term memory, or equivalently the decay to zero of the thermoremanent magnetization, has also been verified (figure 2 of [15]). A similar behaviour has been obtained both for realistic models [12] and experimentally [5, 6].

Second, as a consequence of equation (4.9), the response $\chi(t, t_w)$ introduced in equation (3.2) is given by

$$\chi(t, t_w) = \int_0^{C(t,t_w)} dq' \, X[q'] = F[C(t, t_w)] - F[0] \equiv \tilde{F}[C(t, t_w)]. \tag{8.1}$$

Hence, for large enough times t and t_w, the times (t, t_w) enter parametrically in a plot $\chi(t, t_w)$ against $C(t, t_w)$ and all the points obtained for different pairs (t, t_w) should lie on a *single* universal curve, the integral function $\tilde{F}$ of X.

The plot $\chi(t, t_w)$ against $C(t, t_w)$ for the hypercubic cell of dimension $D = 15$ at temperature $T = 0.2$ is shown in figure 2, together with the second integral of the static $P(q')$ evaluated in $C(t, t_w)$ for the SK model. The curves for different t_w roughly coincide; the departure is not systematic with respect to t_w. However, they do not coincide with the corresponding static curve for the SK model at that temperature. This could be an effect of the finite dimension D, not inconsistent with the static results of [27]. It was found there that the function $P(q')$ for small q' is quite smaller for the hypercubic cell of dimension $D = 12$ than for the SK model.

Third, assumption (iv) on the existence of the triangle relations and its implications can be tested numerically in the following way:

(a) Choose a number $\mathcal{C}$ and a large number t_w.

(b) Determine the time t_{max} such that $C(t_{max}, t_w) = \mathcal{C}$.

(c) Plot, for all times t ($t_w < t < t_{max}$), $C(t_{max}, t)$ against $C(t, t_w)$.

(d) Repeat the procedure for a larger t_w and the same $\mathcal{C}$.

The limiting curve, obtained as t_w becomes larger, is the (implicit) function given by equation (4.12)

$$\mathcal{C} = f(x, y) \tag{8.2}$$

or

$$y = \bar{f}(x, \mathcal{C}). \tag{8.3}$$

The existence of such a limiting curve and its continuity (but not necessarily differentiability) is just the content of assumption (v). Hence, we see that such an assumption is indeed quite plausible since it is difficult to think of a situation in which this does not happen.

If, as has been found in the previous sections, ultrametricity holds, then the area limited by the horizontal line $x \in [\mathcal{C}, 1]$, $y = \mathcal{C}$, the vertical line $x = \mathcal{C}$, $y \in [\mathcal{C}, 1]$ and the curve constructed following the procedure above would vanish when $t_w \to \infty$†. Studying the behaviour of this area is more practical than simply looking at the curves, since its calculation involves many points and reduces the noise. In figure 3, we present the curves obtained in this way (N.B. these curves have been smoothed using a local interpolation in order to better show the qualitative tendency; error bars of order $\simeq 0.01$ should be taken into account). In the inset, we include a log–log plot of the area against t_w. The approach of the curves to their limit is very slow and this could raise the suspicion that this is a finite-size effect. We have checked, however, that for a system four times smaller the decrease in area

† In the p-spin spherical model one expects instead the limiting curve $y = q\mathcal{C}/x$ (cf equation (5.1.8)).

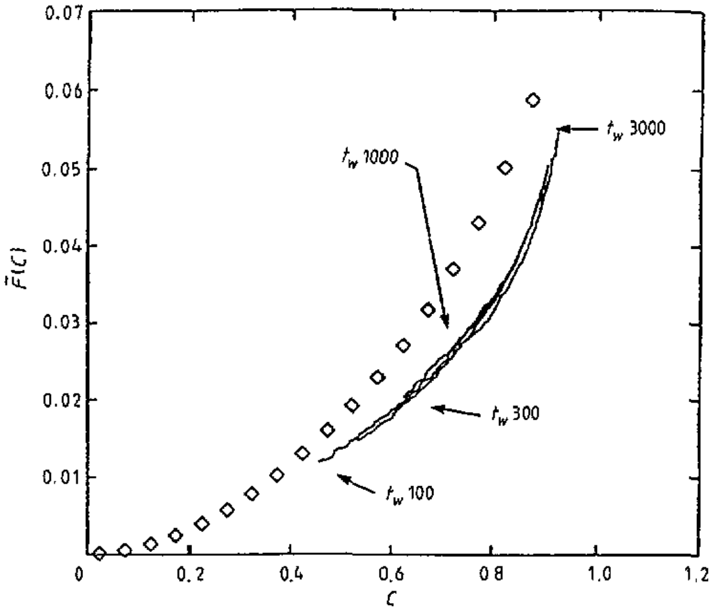

Figure 2. $\chi(t+t_w, t_w)$ plotted against $C(t+t_w, t_w)$. The curves have been obtained by simulating a hypercubic spin-glass cell of dimension $D = 17$ at a subcritical temperature $T = 0.2$ for $t_w = 30, 100, 300, 1000, 3000$. The points represent the static curve of the SK model.

is not very different. The qualitative trend does not depend on the temperature; we have also checked these results for higher subcritical temperatures, though we shall not present them here. Finally, note that the value of q ($\simeq 0.92$ for $T = 0.2$) is easily seen in the figure.

Because the correlation functions, even though easy to compute numerically, are hard to measure in an experimental system, it is convenient to have a relation involving only the easily measurable quantity χ. Consider the function $\mathcal{F}$ defined by

$$\chi(t, t') = \mathcal{F}(\chi(t, t''), \chi(t'', t')). \tag{8.4}$$

From the preceding sections, we have that

$$\mathcal{F}(x, y) = \tilde{F}[f(\tilde{F}^{-1}[x], \tilde{F}^{-1}[y])] \tag{8.5}$$

so that if f is associative (commutative) then $\mathcal{F}$ is also associative (commutative). Indeed, $\mathcal{F}$ and f are isomorphic as a group law [24] and all results quoted for the function f carry through to $\mathcal{F}$. In the ultrametric case, since $\tilde{F}$ is a growing function, we have

$$\mathcal{F}(x, y) = \min(x, y). \tag{8.6}$$

Although the results of the present paper are in principle only valid for mean-field systems, we have not resisted the temptation to check this last relation with the experimental data for spin glasses of [6] with negative results. The fuction $\chi(t, t')$, for experimental systems, follows an almost perfect t'/t law and is, in this sense, more similar to a system with discrete scales.

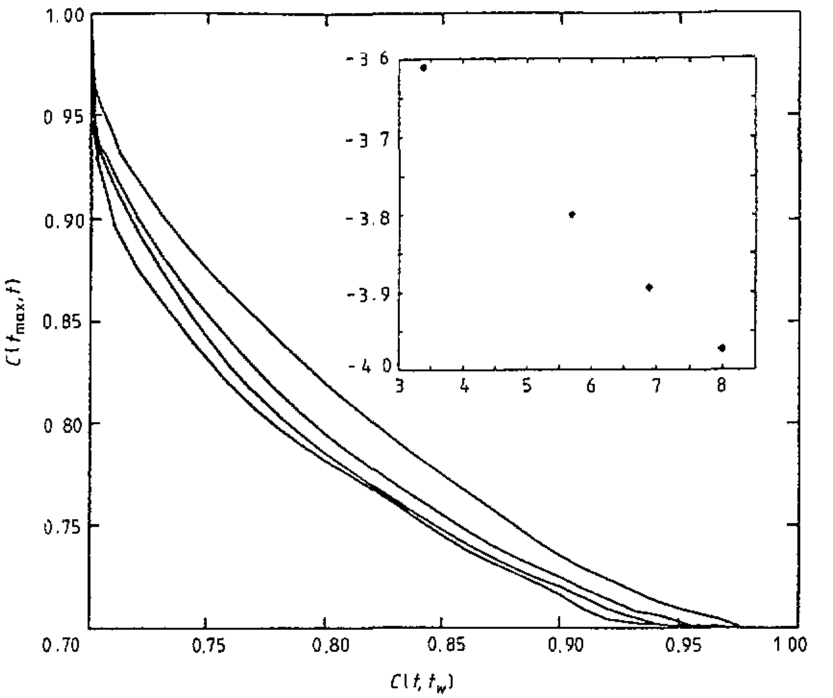

Figure 3. $C(t_{max}, t)$ plotted against $C(t, t_w)$, $t_w \leqslant t \leqslant t_{max}$, for fixed $C(t_{max}, t_w) = 0.7$. $D = 15$ and $T = 0.2$; the four curves correspond to $t_w = 100$, 300, 1000 and 3000. In the inset log–log plot (area against t_w) $D = 15$, $T = 0.2$ and $t_w = 30$, 100, 300, 1000, 3000.

A detailed analysis of the numerical solution of the mean-field dynamical equations corresponding to the relaxation of a particle in a random potential in infinite dimensions was carried out in [16]. In particular, it is verified there that the results for large waiting times coincide with those of the static treatment to order N with great precision. This coincidence is also found here for the SK model, but not in the p-spin spherical model [14]. The common element of the former two models is that the replica analysis of their statics involves an infinite number of breakings of the replica symmetry, while the p-spin model has statically only one breaking and dynamically only one discrete scale apart from the FDT.

An easy test of the correctness of the solution we find within this scenario is to analyse the large-time behaviour of, for instance, the energy and susceptibility. In the case of the SK model, the energy density should relax to the equilibrium value, while in the p-spin spherical model it should relax to a threshold value above the equilibrium value [29].

9. Discussion

We are now in a position to qualitatively discuss the out-of-equilibrium dynamics of this model.

The system first relaxes rapidly to an energy which is slightly above the equilibrium energy. In this region it starts encountering 'traps'. Within a trap, the relaxation is rapid and described by Sompolinsky–Zippelius dynamics [17].

As time passes, the energy relaxes slowly towards the equilibrium value, and the O(N) difference between the dynamical and equilibrium energy goes to zero. The actual states contributing to equilibrium have an energy differing by only O(1) from the lowest state [2] and are not reached in finite times. However, their cousins, the long-time traps, resemble them in their geometry (value of q, relaxation within them) except that the barriers surrounding a true state are divergent, while those surrounding traps are finite, though large. The evolution away from traps becomes slower and slower as time passes so that traps encountered at longer times tend to increasingly resemble the actual states contributing to the equilibrium.

The fact that the system relaxes to values that are equal to order N of the equilibrium values (energy, $P_d(q)$, etc) and that the dynamical and static critical temperatures coincide *is not* a general feature of the spin-glass out-of-equilibrium dynamics but a property of this model. A different situation has been found for the p-spin spherical model [14] in which the non-equilibrium dynamics goes to a threshold level that is above (to O(N)) the states that contribute to the Gibbs–Boltzmann measure (and, hence, $P_d(q) \neq P(q)$ and the asymptotic energy is different from the Gibbs–Boltzmann-measure energy). The reason for this difference can be seen by considering the TAP approach. In the p-spin spherical model, all the TAP valleys below the threshold are separated by infinite barriers and have a positive-definite Hessian. The system should remain trapped within any of these barriers but never does because it remains touring at the threshold energies above which there are no valleys and below which the barriers are O(N), i.e. in the small range of free energies in which the barriers are O(1).

In the SK model, there seems to be no threshold of this kind in the sense that the TAP valleys of the free energy encountered above the equilibrium free energy should be separated by only O(1) barriers. This is quite reasonable if one accepts that most of these solutions (unlike the ones of the p-spin spherical model) are 'born' by division of other solutions as the temperature is lowered and, moreover, their Hessian contains zero eigenvalues [30].

We also note that both the hypotheses of weak ergodicity breaking and of weakness of the long-term memory can be understood within this scenario. Since no traps are true states, the system eventually drifts away, forgetting the characteristics of any given trap (and in particular its magnetization).

10. Conclusions

We have presented the relaxational dynamics of the SK model in the thermodynamic limit in a way that naturally involves an asymptotic out-of-equilibrium regime and ageing effects. We have restricted ourselves to a situation in which the system is rapidly cooled to a subcritical temperature and every external parameter is afterwards left constant, as in some experimental settings. A different approach to the dynamics is to explicitly consider changes in the external parameters such as the temperature [20, 21].

We have shown that the asymptotic dynamical equations can be solved under mild assumptions which we have tried to state as explicitly as possible. Since we are not allowing for the crossing of infinite barriers, all the assumptions can be checked numerically with relatively small computer times and we have presented preliminary results in this direction. We have derived a set of equations containing only the correlation function and the relation between three correlations (triangle relation) and we have found that the unique solution to these equations implies ultrametricity for every three widely separated times. Without the assumption of time homogeneity and in the absence of any external parameter controlling the scales, it is not *a priori* obvious how to define these scales. We have given a precise definition of correlation scales which can be applied to other models.

The present treatment has important common elements with both the static-replica analysis and the Sompolinsky dynamics.

As regards to the former, this is due to an underlying formal algebraic similarity between the replica treatment and the asymptotic dynamics. Quite generally, the asymptotic dynamical equations can be derived using this similarity and their solution has a connection with the replica solution with an ansatz à la Parisi [23] (although this does not necessarily mean that the statics and asymptotic dynamics should give the same results for every model).

It is worthwhile discussing in more detail the similarities and differences with Sompolinsky's dynamics. In that framework, one assumes time homogeneity plus a relation like (4.9) between the correlation and response functions. Then one further assumes that the correlation function relaxes (ever more slowly) to zero for widely separated times (in the absence of a magnetic field). With these assumptions one hopes to have a representation of the equilibrium dynamics after an infinite waiting time and for large but finite N.

A well known problem of this picture is that the decaying to zero of the correlations is incompatible with the equilibrium solution [2] unless one considers multiple dynamical solutions for times long enough as to allow for infinite-barrier crossing [19]. Moreover, there is an additional puzzle: the hypothesis of time homogeneity applied to the p-spin spherical model fails to give the equilibrium values [25].

After the work of Sompolinsky, Ginzburg [11] considered the effect of a perturbation on a spin glass which is already in equilibrium [11]. This is different from our approach since we do not assume that the system has reached equilibrium and we find that it never does. Furthermore, the mechanism for barrier crossing we invoke here is not related to the size of the system but rather to its age.

As mentioned in the introduction, we work here with N infinite, and any infinite time scale arises not because N or any other parameter go to infinity but because it is the very age of the system which imposes the rhythm of the relaxation, which eventually becomes very slow. The assumption that the correlations go to zero in this context is just related to weak ergodicity breaking and the observable ageing effects and does not contradict the statics. Furthermore, since we are considering finite albeit long times, the solution for the correlation and response functions is unique. Yet the fast relaxation for small time differences (but large overall times) is identical to the one found by Sompolinsky–Zippelius with the only change being the reinterpretation of the state as a trap or channel from which the system always escapes.

As noted in section 5, all the results in this paper are invariant with respect to reparametrizations in time. This invariance is not a true property of the dynamics, but is the result of using equations of motion that are, for large time differences, only asymptotically valid. The true solution has to choose one asymptote between all the family of reparametrizations we have obtained. This problem is quite common in the asymptotic matching of the solutions to differential equations. The application of such concepts to the spin-glass problem is still an open question. Many of the most interesting results (decay law of the energy, of the magnetization, etc) will only be available when one will be able to go beyond reparametrization invariance.

There are quite a few open questions, even at the level of reparametrization-invariant results. One would like to have a deeper understanding of the function $X(C)$ and the dynamical generating function $P_d(q)$. It may be that some general theorems can be derived for the asymptotic non-equilibrium regime.

Some of the questions discussed here are quite general and it would be interesting to try them in other models.

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Appendix A

In this appendix, we show that, assuming that $f(x, y)$ is smooth within a discrete scale, the upper limit of the scale acts as the *neutral* (i.e. like the identity in a product) and that the lower limit acts as the *zero* in the product.

The associativity relation (5.1.5) implies

$$\frac{\partial f(x, f(y, z))}{\partial x} = \left.\frac{\partial f(w, z)}{\partial w}\right|_{w=f(x,y)} \frac{\partial f(x, y)}{\partial x}. \tag{A.1}$$

Consider an isolated fixed point a_1^* separating two discrete scales and set $y = z = a_1^*$ in the above equation:

$$\frac{\partial f(x, a_1^*)}{\partial x}\left(1 - \left.\frac{\partial f(w, a_1^*)}{\partial w}\right|_{w=f(x,a_1^*)}\right) = 0. \tag{A.2}$$

This admits the solution

$$f(x, a_1^*) = g(a_1^*) \tag{A.3}$$

since $f(x, a_1^*)$ is continuous and $f(a_1^*, a_1^*) = a_1^*$, $g(a_1^*) = a_1^*$. For $x > a_1^*$, this is a possible solution. For $x < a_1^*$, $f(x, a_1^*) \leqslant x < a_1^*$, so this solution is not admitted in this case.

The other solution to equation (A.2) is

$$\left.\frac{\partial f(w, a_1^*)}{\partial w}\right|_{w=f(x,a_1^*)} = 1. \tag{A.4}$$

Consider the successive fixed point $a_2^* < a_1^*$. $f(x, a_1^*)$ satisfies

$$a_2^* \leqslant f(x, a_1^*) \leqslant a_1^* \tag{A.5}$$

and, because of equation (5.1.2), $f(x, a_1^*)$ is a monotonically increasing function of x. Equality on the left-hand side takes place when $x = a_2^*$ and on the right-hand side when $x = a_1^*$. Hence, for every w within the discrete scale, we can solve $w = f(x, a_1^*)$ for x. Equation (A.4) becomes

$$\frac{\partial f(y, a_1^*)}{\partial y} = 1 \tag{A.6}$$

$\forall\, y \in (a_2^*, a_1^*)$. The solution is $f(y, a_1^*) = y + k(a_1^*)$; since $f(a_1^*, a_1^*) = a_1^*$, then $k(a_1^*) = 0$ and we have finally

$$f(y, a_1^*) = y. \tag{A.7}$$

This solution is only possible if $y < a_1^*$. Otherwise, $y = f(y, a_1^*) \leqslant a_1^*$ and this is incompatible with $y > a_1^*$.

Hence, for x within a discrete scale, $a_2^* < x < a_1^*$, equations (A.3) and (A.7) imply

$$\begin{aligned} f(x, a_2^*) &= a_2^* \\ f(x, a_1^*) &= x \end{aligned} \tag{A.8}$$

and a_2^* and a_1^* are the *zero* and *neutral*, respectively.

Appendix B

In this appendix, we review some results of formal group theory that give the general form of $f(x, y)$ *within a discrete scale*. Let a_1^* and a_2^* be two consecutive fixed points $a_1^* > a_2^*$. We assume that $f(x, y)$ is smooth for $x, y \in (a_2^*, a_1^*]$. From (A.8), we have that a_1^* is the neutral element within this range. Under this assumption, one can show that [24]: $f(x, y)$ is commutative and can be written as

$$f(x, y) = \ell^{-1} \circ (\ell(x) + \ell(y)) \tag{B.1}$$

with $\ell(z)$ given by

$$\ell(z) = -\int_z^{a_1^*} \mathrm{d}z' \left(\left. \frac{\partial f(w, z')}{\partial w} \right|_{w=a_1^*} \right)^{-1} \tag{B.2}$$

$\ell(a_1^*) = 0$. Because we are within a discrete scale, the denominator in the integral is positive definite for $z' \in (a_2^*, a_1^*]$ and it first vanishes in $z' = a_2^*$. The function $\ell(z)$ is increasing and negative semi-definite ($\ell(a_1^*) = 0$).

We can also define

$$\jmath(z) = \exp(\ell(z)) \tag{B.3}$$

to obtain

$$f(x, y) = \jmath^{-1} \circ (\jmath(x) \cdot \jmath(y)). \tag{B.4}$$

Writing (B.1) in terms of the correlations at three times

$$\ell \circ C(t_{\mathrm{max}}, t_{\mathrm{min}}) = \ell \circ C(t_{\mathrm{max}}, t_{\mathrm{int}}) + \ell \circ C(t_{\mathrm{int}}, t_{\mathrm{min}}) \tag{B.5}$$

the crossed second derivative vanishes

$$\frac{\partial^2 \ell \circ C(t_{\mathrm{max}}, t_{\mathrm{min}})}{\partial t_{\mathrm{max}} \partial t_{\mathrm{min}}} = 0. \tag{B.6}$$

The solution to this equation is

$$\ell \circ C(t_1, t_2) = \tilde{h}_1(t_1) - \tilde{h}_2(t_2) \tag{B.7}$$

for some functions $\tilde{h}_1$, $\tilde{h}_2$. Inserting this into (B.5), we see that $\tilde{h}_1(t) = \tilde{h}_2(t) = \tilde{h}(t)$. If we now define $\lambda(t)$ implicitly by

$$C(t, \lambda(t)) = a^* \tag{B.8}$$

where a^* is the largest correlation in the scale, for large t we have

$$\lim_{t\to\infty} \tilde{h}(t) - \tilde{h}(\lambda(t)) = \ell(a^*) = 0. \tag{B.9}$$

Defining $h(t) = \exp(-\tilde{h}(t))$

$$\jmath \circ C(t_1, t_2) = \frac{h(t')}{h(t)} \tag{B.10}$$

with

$$\lim_{t\to\infty} \frac{h(\lambda(t))}{h(t)} = a^*. \tag{B.11}$$

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